Lecture 1

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POISSON AND PRESYMPLECTIC GEOMETRY

(references: section 2.4 of my thesis & Fernandes, Marcut - Lectures)

Definition Proposition Exercise

I) POISSON ALGEBRAS

Poisson algebra

Poisson and Hamiltonian derivations

Morphism of Poisson algebras

Coisotrope

1.1 Reduction of Poisson Algebras

If $I \subset A$ is a coisotrope then

$$N(I)/I =: A'$$

inherits the structure of a Poisson algebra, here

$$N(I) = \{ a \in A \mid \{a, I\} \subseteq I \}.$$

is, by construction, the smallest Poisson subalgebra that contains I as both a multiplicative and Lie ideal. The reduced Poisson structure is then characterized by the fact that the projection map is a Poisson algebra morphism

$$p : (N(I), \cdot, \{,\}) \longrightarrow (A', \cdot', \{,\}').$$

Proof..

II) POISSON MANIFOLDS

Poisson manifold, Poisson map

Poisson and Hamiltonian vector fields / Hamiltonian map

Poisson bivector

Hamiltonian distribution

Coisotropic submanifold

1.2 Characterization of Coisotropic Submanifolds (Prop. 2.4.2 in thesis)

Product Poisson manifold

Coisotropic relation

1.3 Poisson maps as coisotropic relations (Prop. 2.4.3 in thesis)

1.4 Coisotropic Reduction of Poisson manifolds (Prop. 2.4.4 in thesis)

Poisson submanifold

Hamiltonian group action

III PRESYMPLECTIC MANIFOLDS

Presymplectic manifold

Presymplectic map

Characteristic distribution

Hamiltonian vector fields and admissible functions

1.5 The Admissible functions form a Poisson algebra

Proof.

Isotropic submanifold

Hamiltonian group action

1.6 Hamiltonian Presymplectic Reduction (Prop. 2.4.5 in thesis)

IV SYMPLECTIC MANIFOLDS

A note on the distributions associated with Poisson an presymplectic manifolds:

$\boxed{\text{Poisson } (M,\pi)}$

$\pi^\#(T^*P)$ involutive from $[X_f, X_g] = X_{\{f,g\}} \Leftarrow \{,\}$ Jacobi

assume we find integral submanifolds $M \subset P$,

then M is a Poisson submanifold whose bivector is non-degenerate thus defining a non-degenerate 2-form $\pi^{\#\#} \in \Omega^2(M)$ that

$$d\pi^{\#\#} = 0 \quad \Leftarrow \quad \pi \text{ Poisson}$$

then $(M, \pi^{\#\#})$ is a non-degenerate presymplectic manifold.

$\boxed{\begin{array}{c}\text{Presymplectic} \\ (M, \omega)\end{array}}$ $\operatorname{Ker}(\omega^\flat)$ involutive from $d\omega = 0$

assume it is regular with integral foliation $\mathcal{R}$ so that there is a submersion $q : S \to M := S/\mathcal{R}$, by construction we have:

$$C^\infty_\omega(S) = q^* C^\infty(M)$$

The Poisson bracket on $C^\infty_\omega(S)$ induces a Poisson structure on M, let us denote it by (M, π). This is, furthermore, non-degenerate since its musical map is injective

$$\pi^\#(df) = 0 \Leftrightarrow \pi^\#(df)[g] = 0 \quad \forall g \in C^\infty(M)$$

$$\Leftrightarrow \pi(df, dg) = \{f, g\} = \{q^*f, q^*g\}_\omega = 0$$

$$\Leftrightarrow \omega(X_{q^*f}, X_{q^*g}) = 0$$

$$\Leftrightarrow X_{q^*f} \in \operatorname{Ker}(\omega) \Rightarrow$$

$$\Leftrightarrow \omega^\flat(X_{q^*f}) = d\, q^* f = 0$$

$$\Leftrightarrow q^* df = 0 \Rightarrow df = 0$$

Symplectic manifold

Symplectic map

Lagrangian relations, Weinstein's symplectic category